2015 Che2410 – Homework Assignment #5
Due on Dec. 4th at midnight

1. Find solutions that are valid for the following differential equation near $x = 0$,

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}(\alpha + \beta x)y = 0$$

a) Find solutions when $\beta = 0$ and $\alpha \leq 9/8$.

$y(x) = A_0^+ x^{r_+} + A_0^- x^{r_-}$ where $r = \frac{3 \pm \sqrt{9-8\alpha}}{4}$

If $\alpha = \frac{9}{8}$ then $r_+ = r_-$ and so we need a different 2nd linearly independent solution:

$y(x) = A_0^+ x^r + Bx^r \ln x$

b) Use Frobenius theory to obtain the general solution for $\alpha = 1$. Write the power series for $y^+ and\ y^-$ in terms of elementary functions.

Substituting $\alpha = 1$ we get,

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}(1 + \beta x)y = 0$$

which gives us the following coefficients for the Frobenius Theorem:

$R_0 = 1, P_0 = -\frac{1}{2}, Q_0 = \frac{1}{2}, Q_1 = \frac{1}{2}$B

with all higher coefficients being zero. The indicial equation in our case is:

$s(s-1) - \frac{1}{2}s + \frac{1}{2} = 0 \rightarrow s_+ = 1, s_- = \frac{1}{2}$

$s_+ - s_- = \frac{1}{2}$ which is NOT a positive integer, so we know there are two linearly independent power series solutions

Simplifying the general recurrence formula gives us:

$$A_n^+ = \frac{-\beta}{2n^2 + n}A_{n-1}^+, \text{ and } A_n^- = \frac{-\beta}{2n^2 - n}A_{n-1}^-$$

If we play around a bit with the simplified recurrence formula, we can begin to see a pattern:

$$\begin{aligned} A_n^+ &= \frac{-\beta}{2n^2 + n}A_{n-1}^+ \\ &= \frac{-2\beta}{(2n+1)2n}A_{n-1}^+ \\ &= \frac{(-2\beta)(-2\beta)}{(2n+1)2n(2n-1)(2n-2)}A_{n-2}^+ \\ &= \frac{(-2\beta)(-2\beta)(-2\beta)}{(2n+1)2n(2n-1)(2n-2)}A_{n-3}^+ \\ &= \cdots \\ &= \frac{(-2\beta)^n}{(2n+1)!}A_0^+ \end{aligned}$$

Similarly, we get for the A_n^- coefficients, the following non-recurrence formula:

$$A_n^- = \frac{(-2\beta)^n}{(2n)!} A_0^+$$

Then

$$y_+(x) = A_0^+ x \sum_{n=0}^{\infty} \frac{(-2\beta)^n}{(2n+1)!} x^n$$

which can be optionally represented using a sin function as:

$$y_+(x) = A_0^+ \sqrt{\frac{2\beta}{x}} \sin\sqrt{2\beta x}$$

and

$$y_-(x) = A_0^- x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-2\beta)^n}{(2n)!} x^n$$

which can be optionally represented using a cos function as:

$$y_-(x) = A_0^- x^{\frac{1}{2}} \cos\sqrt{2\beta x}$$

c) Once you have the general solution from part (b), consider the limit of $\beta \ll 1$ perform a Taylor expansion in β (i.e., find $y^+ = y_0^+ + \beta y^+$ and similarly for y^-). How do the solutions in this limiting case compare to the solutions in part (a)?

Writing out the first few terms of y_+ give us:

$$y_+(\beta) = A_0^+ x \left(1 - \frac{2\beta}{3!} x + \frac{4\beta^2}{5!} x^2 + \cdots \right)$$

Taking the derivative with respect to β gives us:

$$y'_+(\beta) = A_0^+ x \left(0 - \frac{2}{3!} x + \frac{8\beta}{5!} x^2 + \cdots \right)$$

$At\ \beta = 0$, we get:

$$y_+(\beta = 0) = A_0^+ x, \text{ and } y'_+(\beta = 0) = A_0^+ \tfrac{-2}{3!} x^2$$

So, a Taylor series in β for $y_+(\beta)$ looks like:

$$y_+(\beta) \approx y_+(\beta = 0) + \beta y'_+(\beta = 0) + \frac{\beta^2 y''_+(\beta=0)}{2!} + \cdots$$

Except that we ignore terms that have β^2 because $\beta \ll 1\ldots$ leave us with just the first two terms. Plugging in the above gives us:

$$y_+(\beta) \approx A_0^+ x + \beta A_0^+ \frac{-2}{3!} x^2 = A_0^+ x \left(1 - \frac{2\beta}{3!} x\right)$$

and similarly for $y_-(\beta) = A_0^- x^{\frac{1}{2}}(1 - \beta x)$

Thus, in the limit as $\beta \to 0$, our solutions in this case are exactly the same as in part (a). (Provided one substitutes $\alpha = 1$ into the solution in part (a).

2. Consider the equation:

$$y'' - xy = 0$$

This equation is called Airy's equation. It is important for modeling quantum-mechanical particles hitting walls defined by smooth potentials. Use Frobenius theory to obtain the general solution this equation. The two homogeneous solutions are called Airy functions and cannot be expressed in terms of other elementary functions.

First determine the Frobenius Theorem coefficients:

$$R_0 = 1, P_0 = 0, Q_0 = 0, Q_1 = 0, Q_2 = 0, Q_3 = 1$$

with all higher coefficients being equal to zero.

Our indicial equations results in $s_+ = 1$ and $s_- = 0$. Since $s_+ - s_- = 1$ which is a positive integer, we do not know if we will have a 2nd linearly independent power series solution and will have to check.

However, checking the $n = 1$ case for the A_n^- series is straightforward because R_1, P_1, and Q_1 are all zero. Therefore $A_1^-(0) = A_0^-(0 + 0 + 0)$ is satisfied, and A_1^- is arbitrary and can be set to zero.

Thus, we know we will have two linearly independent power series solutions.

The simplified recurrence formula for A_n^+ is:

$$A_n^+ = \frac{1}{(n+1)n} A_{n-3}^+$$

and similarly,

$$A_n^- = \frac{1}{n(n-1)} A_{n-3}^-$$

writing out the non-recurrence formula in this case is tricky, because it is difficult to describe the pattern concisely. Any attempt to describe the pattern more concisely than in the regular recurrence formula is given full credit.

The general solution is thus: $y(x) = x \sum_{n=0}^{\infty} A_n^+ x^n + \sum_{n=0}^{\infty} A_n^- x^n$ using the above recurrence formulas for the coefficients.

3. Determine the two values of the constant α for which all solutions of

$$xy'' + (x-1)y' - \alpha y = 0$$

can be written as a power series (i.e., $y = x^s \sum_{n=0}^{\infty} A_n x^n$).

To apply Frobenius theory, first multiply the equation by x to get it into the right form.

Then compute the R-P-Q coefficients:

$$R_0 = 1, P_0 = 1, P_1 = 1, Q_0 = 0, Q_1 = -\alpha$$

The indicial equation leads to $s_+ = 2$ and $s_- = 0$ so $s_+ - s_- = 2$ which is a positive integer, meaning we will need to test the $n = 2$ case for the A_n^- coefficients.

Testing the $n = 2$ case leads to:

$$A_2^-(0) = A_1^-(1-\alpha)$$

Since $A_1^- = -\alpha A_0^-$ we get:

$$A_2^-(0) = A_1^-(1-\alpha) = -A_0^-\alpha(1-\alpha)$$

In order for this equation to be satisfied for any A_0^-, we require that the RHS be zero, and so α must be either 0 or 1.