2015 Che2410 – Homework Assignment #3
Due on Oct. 23rd before midnight

1. (30 pts) Consider the heat (diffusion) equation:

$$\frac{\partial T}{\partial t} = \mathrm{K}\frac{\partial^2 T}{\partial x^2}, \qquad K = 5.0$$

$$T(t,0) = 50\sin(t) + 300, \qquad T(t,10) = 300, \qquad 0 \le t \le 4\pi$$
$$T(0,x) = 300, \qquad 0 \le x \le 10$$

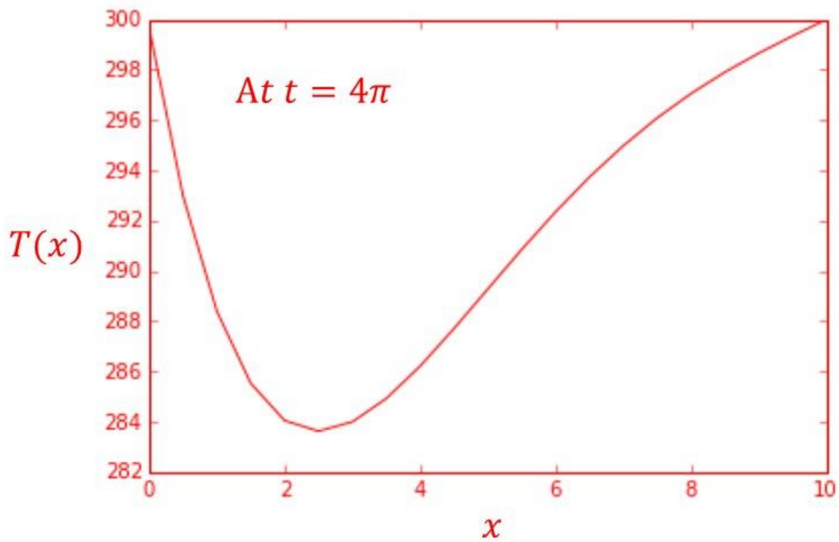
We are modelling the temperature profile of a long and narrow reactor, where the temperature at one end is room temperature, and at the other end the temperature is fluctuating ± 50 degrees around room temperature.

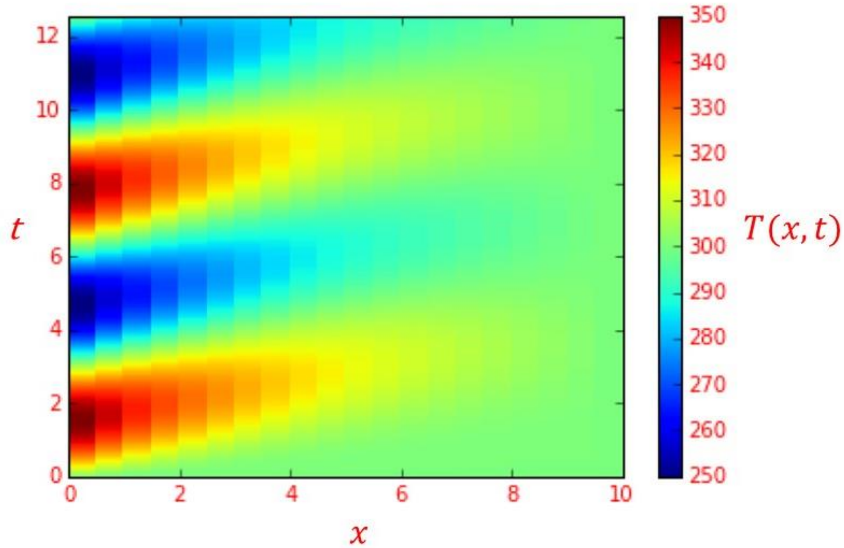
(a) Turn this differential equation into a difference equation by using the Forward Euler method (i.e., forward differencing in time, central differencing in space).

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = K\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}$$

(b) Solve the difference equation using the scheme in (a). Use an extra grid point on each end of the x-domain to handle the boundaries (i.e., do not use periodic boundary conditions). Plot the temperature profile of the entire reactor at $t = 4\pi$. Also, show your results for the entire time-domain and x-domain by representing the T variable by color: create a 2D plot with x on the horizontal axis, t on the vertical axis, and represent T by color (where blue is coldest and red is warmest). See example below.

For $\Delta x = 0.5$ and $\Delta t = 0.01$ the plots looks like:



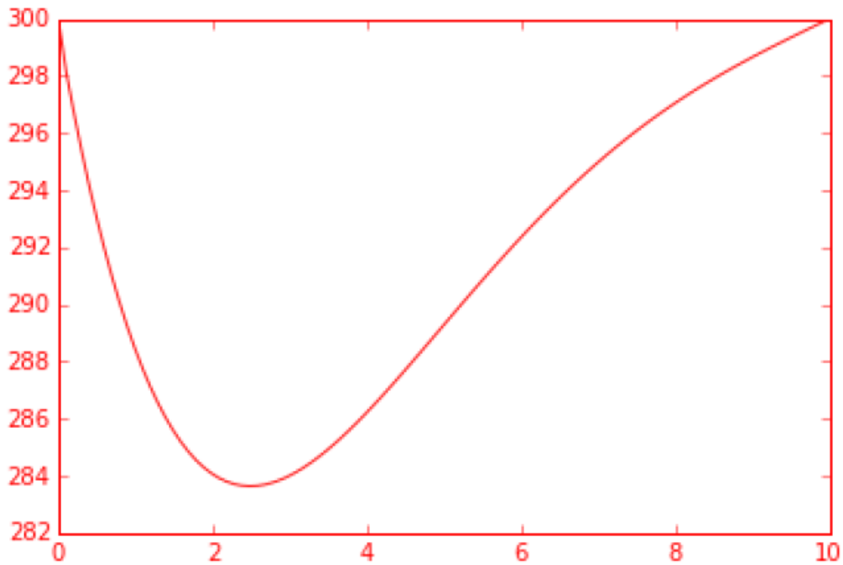


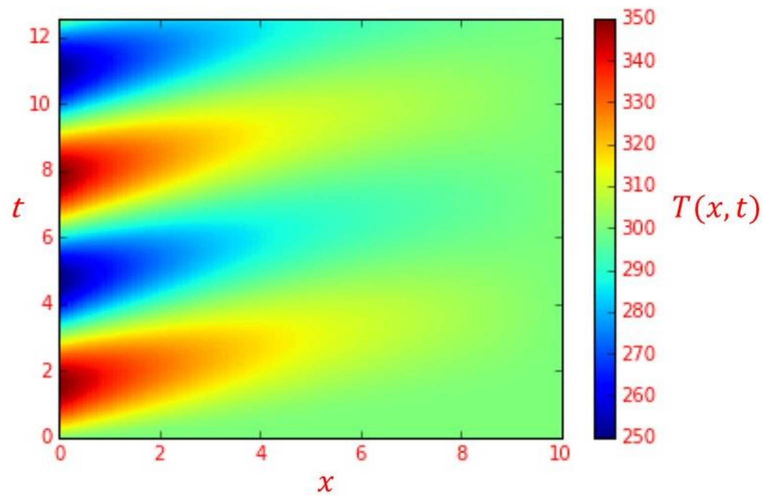
(c) Turn this differential equation into a difference equation by using the Crank-Nicolson scheme.

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = \frac{K}{2}\left(\frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{\Delta x^2} + \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}\right)$$

(d) Solve the difference equation above with the same boundary conditions as in (b). Also report your results using the same plots as in (b).

With Crank-Nicolson, it is easier to get smoother results with less error because of the unconditional stability property.





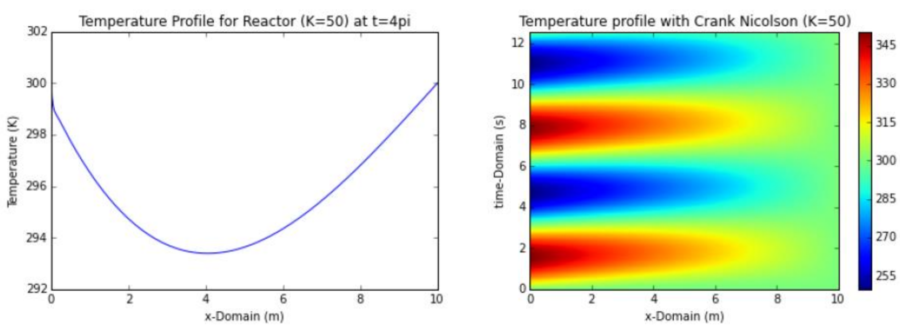
(e) How do your results compare using the two different methods?

Credit was given in this question to students that tried several different Δx and Δt values for both methods, compared the results, and discussed them.

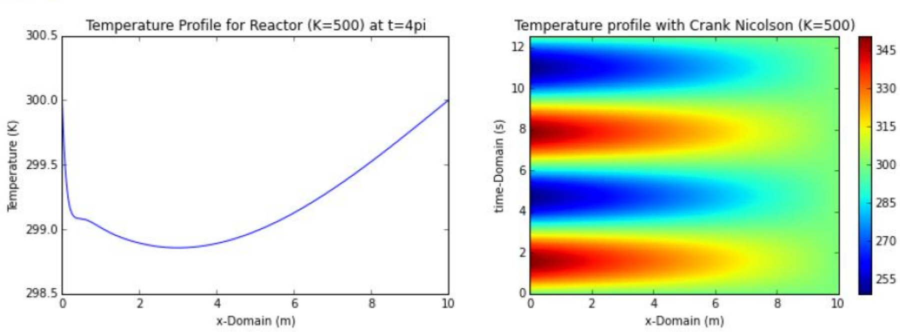
(f) For either method, plot your results also for $K = 50$ and $K = 500$. What changes, what does not change, and why?

Below is a result from one of the students who answered correctly.

K = 50



K = 500



There is a significant difference in the temperature profile when K changes from 5 to 50, but not so much from 50 to 500. K is the thermal diffusivity in this problem, and once diffusivity is sufficiently high, the heat transfers almost instantaneously across the reactor. In this limit, the temperature profile at any given time is just a straight line between the temperatures at the boundaries.