2015 Che2410 – Homework Assignment #2
Due on Oct. 6th at 4:30pm

1. (10 pts) Solve the following differential equation numerically:

$$f_t = \sin(t), \qquad f(0) = 1, \qquad 0 \le t \le 5$$

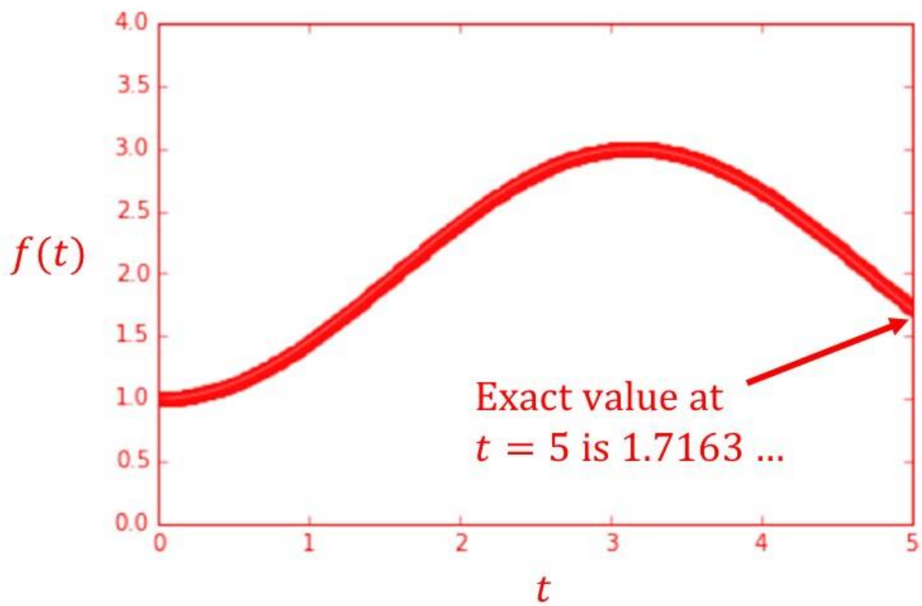
a) Find the exact solution.

Exact solution: $f(t) = 2 - \cos(t)$, $f(t = 5) = 1.7163$

b) Choose a scheme so that your solution at $t = 5$ is within 1% of the exact answer. Plot your numerical solution vs. the exact solution and report the value of both at $t = 5$.

Forward differencing is the simplest scheme to implement here:

$f_{n+1} = f_n + \Delta t \sin(n\,\Delta t)$, where $\Delta t = 0.05$ or less is sufficient.



c) Repeat part (b) but change the timestep to reduce the error by a factor of 10.

Reducing the time step by a factor of 10 will reduce the error by a factor of 10 because forward differencing is a first order method. For people who used other schemes the factor would be different.

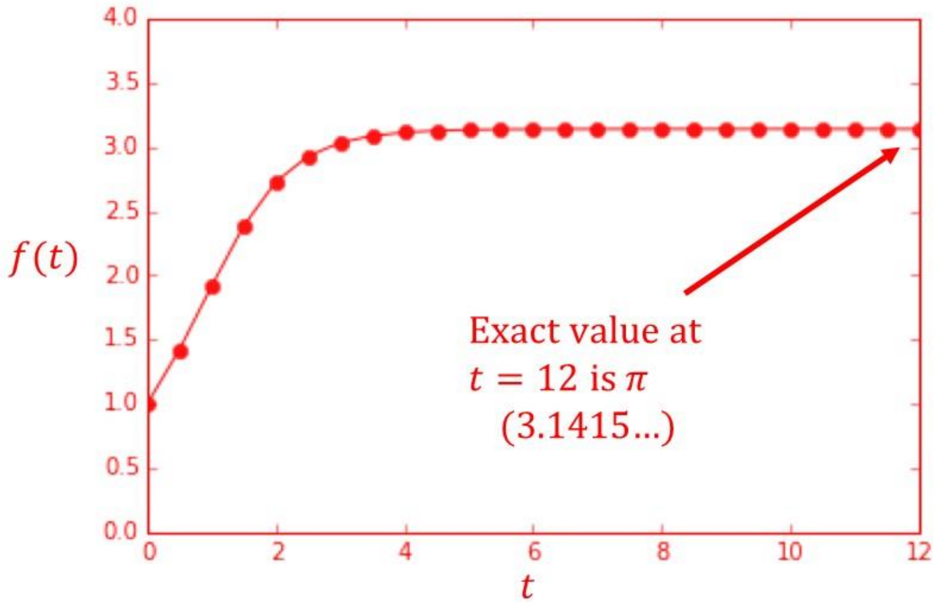
2. (20 pts) Solve the following differential equation numerically. Note that this is a *nonlinear* differential equation – none of the techniques for solving differential equations analytically in this class will work on this equation.

$$f_t = \sin(f), \qquad f(0) = 1, \qquad 0 \le t \le 12$$

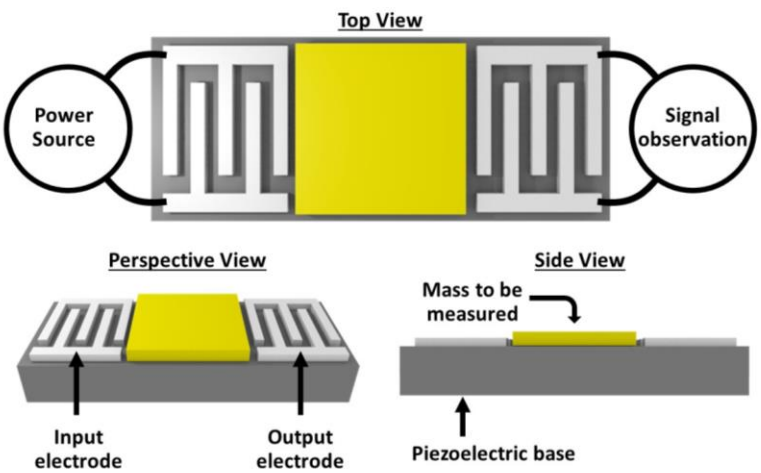
Choose a scheme to solve this equation numerically. Since it may not be possible to compare your answer against an exact answer, keep making your timestep smaller by factors of 2 until your solution at $t = 12$ doesn't change by more than 1%. Plot your best numerical solution and report the value at $t = 12$.

Forward differencing can be used again. The equation looks very similar to the one above:

$f_{n+1} = f_n + \Delta t \sin(f_n)$, with a $\Delta t = 0.5$ you get the plot below:



3. (30 pts) Consider a surface acoustic wave sensor.



In these sensors, a wave is generated at the input electrode, which travels through a sensor material (yellow) and arrives at the output electrode. Depending on the mass of the sensor material, the wave will travel more quickly or more slowly. This traveling wave phenomenon can be modeled by the following partial differential equations:

$$f_t = -0.5\, f_x, \quad |x| < 5, \quad -10 < x < 10, \text{ Sensing material region}$$

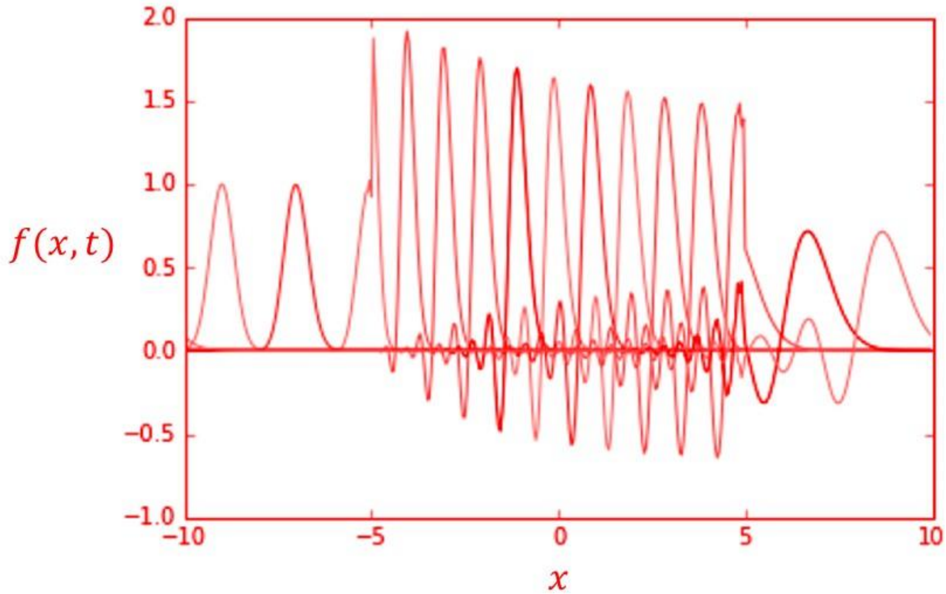
$$f_t = -f_x, \quad |x| \geq 5, \quad -10 < x < 10, \text{ Input/output electrode region}$$

(continued from #3 above)

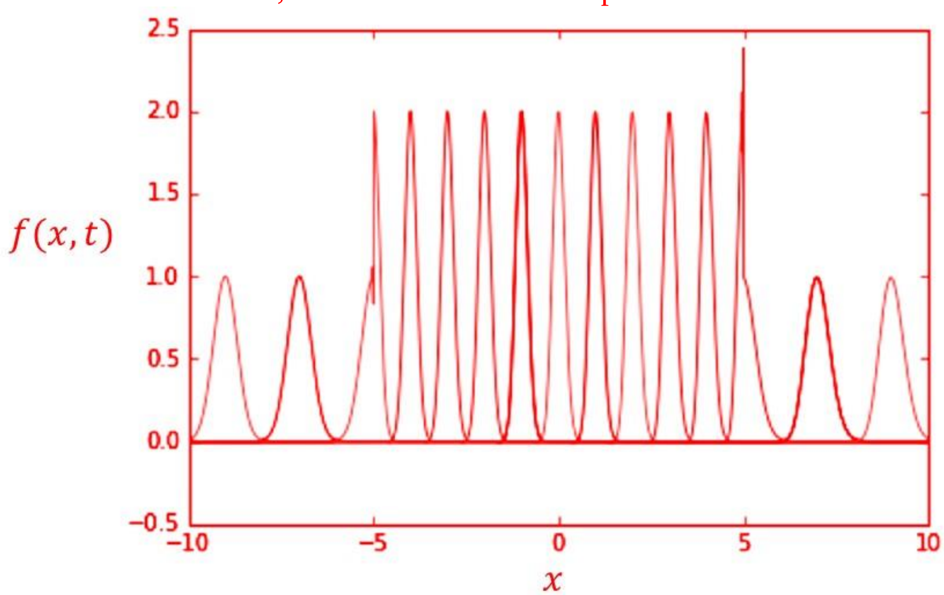
Solve these partial differential equations numerically using the Lax-Wendroff scheme. Assume the wave has the form of a Gaussian with $\sigma = 5$ and that at $t = 0$ the wave is centered at $x = -9$. Apply periodic boundary conditions (even though it is unrealistic) to handle the edges.

a) After finding an appropriate Δt and Δx (justify your choice), plot the solution at various points in time from when it is on the input electrode to when it arrives at the output electrode.

Using $\Delta t = 0.05$ and $\Delta x = 0.01$ gives the following plot (each peak corresponds to the same travelling wave but at different times):



When we choose a smaller time and space step, $\Delta t = 0.01$ and $\Delta x = 0.005$ we get a much cleaner solution. When we make the steps even smaller, the solution looks the same, so we conclude that the plot below is the “real” solution.



b) Describe what happens to the wave in the different regions: How is the peak changing? How is the width changing? How is the wave speed changing?

The wave with a peak height of 1 travels at constant velocity (to the right) until it hits the sensing material boundary, at which point the peak height doubles, the width shrinks by half, and the velocity decreases by a factor of two. The wave travels more slowly throughout the sensing material region, and then reverts back to the input wave form when it passes the $x = 5$ boundary.

Notes: There is no notion of "frequency" in this problem. There is a single travelling wave that moves from left to right.

c) What aspects of your solution are probably "real" vs. what parts are probably due to approximation error?

From comparing the two plots above, we can see two features that disappear when use smaller time/space steps: the small trailing "ripples" behind the travelling wave as it moves far from its initial position (this is dispersion error), and a steadily decreasing peak height (this is dissipation error).

The fact that parts of the solution move across the periodic boundary is not an approximation error… it may be unphysical but it is not due to any approximations and would not go away no matter how small the time/space step.